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Unified Description of Early Universe with Bulk Viscosity

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A class of homogeneous and isotropic zero-curvature Robertson–Walker models with bulk viscosity is studied. Solutions are obtained with the parameter gamma of the "gamma-law" equation of state $p = (\gamma - 1)\rho$ in which the adiabatic parameter γ varies continuously as the universe expands. A unified description of the early evolution of the universe is presented with constant bulk viscosity and time-dependent bulk viscosity in which an inflationary period is followed by a radiation-dominated phase. We also establish the time dependence of the cosmological constant in terms of varying γ index. Some physical properties of the models are also discussed.

1. INTRODUCTION

The simplest model of the expanding universe is well represented by the Friedmann–Robertson–Walker (FRW) models, which are both spatially homogeneous and isotropic. FRW models are in some sense good global approximations of the present-day universe, but it is unreasonable to assume that the regular expansion predicted by these models is also suitable for describing the early stages of the universe. The evolution of the universe is described by Einstein's equations together with an equation of state for the matter content. The history of the universe may be divided int three main periods: (a) the inflationary period, (b) the radiation-dominated period, when there was matter at a very high temperature so that it behaved like isotropic radiation, and (c) the matter-dominated period in which we are at present. Israelit and Rosen (1989, 1993) described these periods using an equation of state in which pressure varies continuously from $-\rho$ to its value during the radiation era ($p = \rho/3$). Recently Carvalho (1996) studied a spatially

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homogeneous and isotropic cosmological model of the universe in which the parameter gamma of the “gamma-law” equation of state $p = (\gamma - 1)\rho$ varies continuously as the universe expands, and presented a unified description of the early evolution of the universe in which an inflationary period is followed by a radiation-dominated phase.

The role of the bulk viscosity $\mu(t)$ in the cosmic evolution, especially as its early stages, seems to be significant. The general criterion for bulk viscosity was given by Weinberg (1972). He pointed out that bulk viscosity might be of importance when considering relativistic and nonrelativistic particles. The homogeneity and isotropy of the universe might have passed through dissipative viscuous phases during its evolution, still retaining spatial symmetries characterized by Friedmann models. Heller (1973) introduced bulk viscosity in the frame of ordinary Friedmann cosmology under a highly idealized assumption of constant coefficient of bulk viscosity. Johri and Sudharsan (1988) investigated the effect of bulk viscosity on the evolution of Friedmann models and found that the presence of a tiny time-dependent component of bulk viscosity would play a crucial role in driving the present-day universe into a steady state.

In this paper, we consider a model with bulk viscosity to study the evolution of the universe as it goes from an inflationary phase to a radiation-dominated era. We apply the gamma-law equation of state in which the parameter γ varies continuously as the universe expands. The paper is organized as follows. In Section 2 we present the basic equations governing the models. In Section 3 we obtain the solution of the field equations for two cases (i) $\mu(t) = \mu_0 = \text{const}$ and (ii) $\mu = \mu(t)$. In Section 4 we establish the cosmological constant in terms of varying γ index. The main conclusions are given in Section 5.

2. THE MODEL

In order to build up cosmological models, we assume that the universe is spatially homogeneous and isotropic with a geometry determined by the Friedman–Robertson–Walker line element

$$ds^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2\, d\theta^2 + \tau^2 \sin^2\theta\, d\phi^2\right) \qquad (1)$$

where $R(t)$ is the scale factor and $k = 0, \pm 1$ is the curvature parameter. If the universe is filled with a cosmic fluid, the energy-momentum tensor T_{ij} is given by

$$T_{ij} = [\rho + (p - \mu\theta)]u_i u_j - (p - \mu\theta)g_{ij} \tag{2}$$

where ρ is the energy-density, p the pressure, μ the coefficient of bulk viscosity, θ the expansion scalar, and u^i the four-velocity vector.

In comoving coordinates, the surviving components of Einstein's field equations

$$G_{ij} = -8\pi G T_{ij} \tag{3}$$

are then

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8}{3}\pi G\rho \tag{4}$$

$$2\frac{\ddot{R}}{R} = -\frac{8}{3}\pi G[\rho + 3p - 3\mu\theta] \tag{5}$$

where an overdot denotes time derivative. Equations (4) and (5) can be rewritten as

$$\frac{\theta^2}{9} + \frac{k}{R^2} = \frac{8}{3}\pi G\rho \tag{6}$$

$$\frac{\dot{\theta}}{3} + \frac{\theta^2}{9} = -\frac{4}{3}\pi G[\rho + 3p - 3\mu\theta] \tag{7}$$

where $\theta = 3\dot{R}/R$.

In order to solve the above field equations, we assume that the pressure p and the energy density ρ are related through the "gamma-law" equation of state

$$p = (\gamma - 1)\rho \tag{8}$$

where the adiabatic parameter γ varies continuously as the universe expands. Recently Carvalho (1996) assumed a scale-dependent γ of the form

$$\gamma(R) = \frac{4}{3}\frac{A(R/R_*)^2 + (a/2)(R/R_*)^a}{A(R/R_*)^2 + (R/R_*)^a} \tag{9}$$

where A is a constant and a is free parameter lying in the interval $0 \le a < 1$. The function $\gamma(R)$ is such that we have an inflationary phase when the scale factor R is less than a certain reference value R_* (i.e., $R << R_*$). As $R(t)$ increases, γ also increases to reach the value 4/3 for $R >> R_*$, when we have a radiation-dominated phase.

Substituting equation (8) into (7), we get

$$\frac{\dot{\theta}}{3} + \frac{\theta^2}{9} = -\frac{8}{3}\pi G\left(\frac{3}{2}\gamma - 1\right)\rho + 4\pi G\mu\theta \tag{10}$$

Eliminating ρ between (6) and (10), we get the first-order differential equation

$$\dot{\theta} + \frac{\gamma}{2}\theta^2 + 3\left(\frac{3}{2}\gamma - 1\right)\frac{k}{R^2} - \alpha\mu(t)\theta = 0 \tag{11}$$

where $\alpha = 12\pi G$. Equation (11) can be rewritten in the form

$$\theta\theta' + \frac{3}{2}\gamma\frac{\theta^2}{R} + 9\left(\frac{3}{2}\gamma - 1\right)\frac{k}{R^3} - 3\alpha\frac{\mu(t)\theta}{R} = 0 \tag{12}$$

where a prime denotes derivative with respect to R. For $k = 0$, equation (12) becomes

$$\theta' + \frac{3}{2}\gamma\frac{\theta}{R} - 3\alpha\frac{\mu(t)}{R} = 0 \tag{13}$$

3. SOLUTION OF THE FIELD EQUATIONS

Case (i): $\mu(t) = \mu_0 =$ const. Substituting equation (9) into (13) and integrating, we obtain

$$\theta = \frac{C + 3\alpha_1[(A/2)(R/R_*)^2 + (1/a)(R/R_*)^a]}{[A(R/R_*)^2 + (R/R_*)^a]} \tag{14}$$

where C is the integration constant and $\alpha_1 = \alpha\mu_0$. When we take $\theta = \theta_*$ for $R = R_*$, a relation between A and C can be written in the form

$$C = [1 + A]\theta_* - 3\alpha_1[A/2 + 1/a], \qquad a \neq 0 \tag{15}$$

The solutions in terms of scale factor R can be obtained by integrating equation (14) as

$$(A/2)(R/R_*)^2 + (1/a)(R/R_*)^a = \frac{C_1 e^{\alpha_1 t} - C}{3\alpha_1} \tag{16}$$

where C_1 is another integration constant. If we adjust C_1 and C such that $C_1 = C = B$, where B is constant, the above equation reduces to

$$(A/2)(R/R_*)^2 + (1/a)(R/R_*)^a = B\frac{e^{\alpha_1 t} - 1}{3\alpha_1} \tag{17}$$

For $R << R_*$, the second term on the left-hand side of equation (17) dominates, which has a phase of exponential inflation and scale factor given by

$$R = \left[aB \frac{e^{\alpha_1 t} - 1}{3\alpha_1} \right]^{1/a} R_* \tag{18}$$

The expansion scalar is given by

$$\theta = \frac{3\alpha_1}{a} [1 - e^{-\alpha_1 t}]^{-1} \tag{19}$$

and the energy-density has the form

$$\rho = \frac{9}{2} \frac{\alpha \mu_0^2}{a^2} [1 - e^{-\alpha_1 t}]^{-2} \tag{20}$$

In the limiting case $a = 0$ at $R = 0$, both ρ and θ are infinite.

For $R >> R_*$, the first term on left-hand side of equation (17) dominates. Then the solution for scale factor is given by

$$R = \left[\frac{2B}{A} \frac{e^{\alpha_1 t} - 1}{3\alpha_1} \right]^{1/2} R_* \tag{21}$$

The expansion scalar and energy-density for this period have the following form

$$\theta = \frac{3\alpha_1}{2} [1 - e^{-\alpha_1 t}]^{-1} \tag{22}$$

$$\rho = \frac{9}{8} \alpha \mu_0^2 [1 - e^{-\alpha_1 t}]^{-2} \tag{23}$$

When we study the solution in the limit $\alpha_1 \to 0$, i.e., $\mu_0 \to 0$, equation (17) gives the solution

$$(A/2)(R/R_*)^2 + (1/a)(R/R_*)^a = \frac{B}{3} t \tag{24}$$

and this case is the perfect fluid model considered by Carvalho (1996).

Case (ii): $\mu = \mu(t)$. In this case we assume the functional form of $\mu(t)$ as

$$\mu(t) = \beta \rho^{1/2} \tag{25}$$

where β is a positive constant. Substituting equation (25) into (13) and using equation (6), we obtain the first-order differential equation

$$\theta' + \left(\frac{3}{2} \gamma - \beta_1 \right) \frac{\theta}{R} = 0 \tag{26}$$

where $\beta_1 = 3\beta(\alpha/2)^{1/2}$. On integration, the expansion scalar is given by

$$\theta = \frac{C_2 R^{\beta_1}}{A(R/R_*)^2 + (R/R_*)^a} \tag{27}$$

where C_2 is the integration constant. If we take $\theta = \theta_*$ for $R = R_*$, the relation between A and C_2 can be written as

$$C_2 = [1 + A]\frac{\theta_*}{R_*^{\beta_1}} \tag{28}$$

An expression for t in terms of scale factor can be obtained by integrating equation (27)

$$C_2 t = 3\left[\frac{A}{R_*^2}\frac{R^{2-\beta_1}}{2-\beta_1} + \frac{1}{R_*^a}\frac{R^{a-\beta_1}}{a-\beta_1}\right. \tag{29}$$

For $R << R_*$, the second term on right-hand side of equation (29) dominates, which has a phase of power law inflation and a solution for the scale factor given by

$$R = \left[\frac{(a-\beta_1)C_2}{3} tR_*^a\right]^{1/(a-\beta_1)} \tag{30}$$

The expansion scalar is given by

$$\theta = \frac{3}{a-\beta_1} t^{-1} \tag{31}$$

and the energy-density has the form

$$\rho = \frac{3}{8\pi G(a-\beta_1)^2} t^{-2} \tag{32}$$

which shows that energy-density varies inversely proportional to the square of the age of universe, and that as $t \to \infty$, the density becomes zero.

When we consider the radiation-dominated phase ($R >> R_*$), the first term on the right-hand side of equation (29) dominates; then the solution for the scale factor is given by

$$R = \left[\frac{(2-\beta_1)C_2}{3A} tR_*^2\right]^{1/(2-\beta_1)} \tag{33}$$

The expansion scalar for this phase is given by

$$\theta = \frac{3}{2-\beta_1} t^{-1} \tag{34}$$

and the density has the form

$$\rho = \frac{3}{8\pi G} \frac{1}{(2 - \beta_1)^2} t^{-2} \tag{35}$$

which shows that the energy density varies inversely proportional to square of the age of universe.

4. TIME-DEPENDENT COSMOLOGICAL CONSTANT

The cosmological constant Λ has attracted the attention of many cosmologists for various reasons. The nontrivial role of the vacuum in the early universe generates a Λ term in the Einstein field equations that leads to the inflationary phase. The inflationary cosmology postulates that during an early universe phase, the vacuum energy was a large cosmological constant. Therefore, in view of the smallness of the cosmological constant observed at present, it is natural to assume that the cosmological constant Λ is a variable dynamic degree of freedom which, being initially very large, has relaxed to its small present value in an expanding universe. The idea of dynamically decaying constant with cosmic expansion has been considered by several authors (e.g., Kalligas *et al.*, 1992; Beesham, 1986).

In this section, by introducing an effective adiabatic index γ, we shall find the time-dependent cosmological constant with constant bulk viscosity as a function of the scale factor.

The Einstein field equations (6) and (11) with a Λ term are, respectively,

$$\frac{\theta^2}{9} + \frac{k}{R^2} = \frac{8}{3}\pi G\rho + \frac{1}{3}\Lambda \tag{36}$$

$$\dot{\theta} + \frac{1}{2}\gamma_0\theta^2 + 3\left(\frac{3}{2}\gamma_0 - 1\right)\frac{k}{R^2} - \frac{3}{2}\gamma_0\Lambda = \alpha\mu(t)\theta \tag{37}$$

Here γ_0 is the constant asymptotic limit of the adiabatic index, which in our case is the limiting value of γ for $R >> R_*$ during the radiation era, that is, 4/3.

We take Λ in the convenient form

$$\Lambda = 8\pi G\rho\lambda \tag{38}$$

Then equation (36) becomes

$$\frac{\theta^2}{9} + \frac{k}{R^2} = \frac{8}{3}\pi G\rho(1 + \lambda) \tag{39}$$

Substituting the value of Λ given by (38) into (37) and using (39) to eliminate ρ, we obtain

$$\dot{\theta} + \frac{1}{2}\frac{\gamma_0}{1+\lambda}\theta^2 + 3\left(\frac{3}{2}\frac{\gamma_0}{1+\lambda} - 1\right)\frac{k}{R^2} - \alpha\mu(t)\theta = 0 \tag{40}$$

Comparing the above equation with (11), we find

$$\gamma = \frac{\gamma_0}{1+\lambda} \tag{41}$$

Now, combining this expression with (38) and (39), we obtain

$$\Lambda = 3\left(\frac{\theta^2}{9} + \frac{k}{R^2}\right)\left(1 - \frac{\gamma}{\gamma_0}\right) \tag{42}$$

Combining equations (9) and (14), we obtain

$$1 - \frac{\gamma}{\gamma_0} = \left(1 - \frac{a}{2}\right)\frac{(R/R_*)^a\theta}{C + 3\alpha_1(1/a)(R/R_*)^a}, \qquad a \neq 0 \tag{43}$$

Substituting this expression in equation (42) for $k = 0$, we get

$$\Lambda = \frac{1}{3}\left(1 - \frac{a}{2}\right)\frac{(R/R_*)^a\theta^3}{C + 3\alpha_1(1/a)(R/R_*)^a}, \qquad a \neq 0 \tag{44}$$

For the time-dependent bulk viscosity as considered in case (ii) of Section 3, we find a similar result by combining equations (9) and (27) for the cosmological constant. For $a > 0$, we have

$$1 - \frac{\gamma}{\gamma_0} = \left(1 - \frac{a}{2}\right)\frac{(R/R_*)^a\theta}{R^{\beta_1}C_2} \tag{45}$$

Substituting this expression in equation (42) for $k = 0$, we get

$$\Lambda = \frac{1}{3}\left(1 - \frac{a}{2}\right)\frac{(R/R_*)^a\theta^3}{R^{\beta_1}C_2} \tag{46}$$

5. CONCLUSION

We have presented homogeneous and isotropic cosmological models with bulk viscosity in which the adiabatic parameter γ of the gamma equation of state varies continuously as a function of scale factor $R(t)$. We studied the transition from the inflationary phase ($R << R_*$) to the radiation-dominated phase ($R >> R_*$) of the universe. For our models the parameter a in equation (9) lies in the range $0 < a < 1$. When we study the solution in the limit $a \to 0$ with initial value $R = 0$, the expansion factor become infinite for constant bulk viscosity and zero for time-dependent bulk viscosity. From equations (31) and (34) we see that expansion scalar θ tends to zero as $t \to \infty$. Models are singularity-free, as the energy density is always finite.

We have also pointed out the similarity between the dynamical behavior of our models with those models which incorporate a time-dependent cosmological constant. Writing the field equations with a time-dependent cosmological constant, we have calculated its value as a function of scale factor.

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